Roll No. ....

E–310

M. A./M. Sc. (First Semester) EXAMINATION, Dec.-Jan., 2020-21

MATHEMATICS

Paper Second

(Real Analysis—I)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. The sequence $\{x^n\}$ where $0 \leq x \leq 1$:
   (a) converges uniformly on [0, 1].
   (b) does not converge uniformly on [0, 1], 0 is the point of non-uniformly convergence.
   (c) does not converge uniformly on [0, 1], 1 is the point of non-uniformly convergence
   (d) None of the above

2. The series $\frac{\cos x}{1^p} + \frac{\cos 2x}{2^p} + \frac{\cos 3x}{3^p} + \ldots\ldots + \frac{\cos nx}{n^p} + \ldots\ldots$ converge uniformly on $\mathbb{R}$ for :

   (a) $p \leq 1$

   (b) $p \geq 1$

   (c) $p < 1$

   (d) $p > 1$

3. If $f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$, $0 \leq x \leq 1$, $n = 1, 2, 3, \ldots$, then :

   (a) the sequence $\{f_n\}$ is uniformly bounded on [0, 1]

   (b) every subsequence of $\{f_n(x)\}$ can converge uniformly on [0, 1]

   (c) Both (a) and (b)

   (d) None of the above

4. Consider the following statements :

   (I) Let $\sum_{n=1}^{\infty} f_n$ be a series of real valued functions on a set E in metric space. If the sum function of the series is continuous on E, then the series converges uniformly on E.

(II) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions on a set E in metric space. If the limit function of the sequence is continuous on E, then the sequence converged uniformly on E.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

5. The power series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$:

(a) converges (absolutely) for all values of x.

(b) does not converges for any values of x (other than 0).

(c) converges for $-1 < x < 1$

(d) None of the above

6. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$ is :

(a) 0

(b) 1

(c) ∞

(d) None of the above

7. The interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$$ is :

(a) $\left(\frac{5}{2}, \frac{7}{2}\right)$

(b) $\left[\frac{5}{2}, \frac{7}{2}\right)$

(c) $\left(\frac{5}{2}, \frac{7}{2}\right]$

(d) $\left[\frac{5}{2}, \frac{7}{2}\right]$

8. Sum of the series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots\ldots$

is :

(a) log 2

(b) $\frac{1}{2} \log 2$

(c) $\frac{2}{3} \log 2$

(d) $\frac{3}{2} \log 2$

9. If $A \in L(R^n, R^m)$ and if $x \in R^n$, then :

(a) $A'(x) = A$

(b) $A''(x) = A$

(c) $A'(x) = I$

(d) None of the above

10. If, $A, B \in L(R^n)$ then :

(a) $BA(x) = A(Bx)$

(b) $BA(x) = B(A\,x)$

(c) $BA(x) = A(B\,x) = B(A\,x)$

(d) None of the above

11. Let $X \subset R^n$ be a vector space with X = {0}, then dim X is :

(a) 0

(b) 1

(c) n

(d) None of the above

12. Let m be a positive integer. If a vector space X is spanned by a set of m vectors, then :

(a) $\dim X \leq m$

(b) $\dim X = m$

(c) $\dim X \geq m$

(d) None of the above

13. Let f be a real function defined in an open set $E \subset R^n$, then $D_{ij} f = D_{ji} f$ if :

(a) $f \in C''(E)$

(b) $f \in C'(E)$

(c) $f \in C(E)$

(d) none of the above

14. A linear operator A on R^n is invertible if and only if :

(a) det [A] $\neq$ 0

(b) det [A] = 0

(c) det [A] = 1

(d) None of the above

15. If $u = x + y + z, \quad v = x - y + z, \quad w = x^2 + y^2 + z^2 - 2xyz,$ then $\frac{\partial(u,v,w)}{\partial(x,y,z)} =$:

(a) 0

(b) $4x$

(c) 4

(d) None of the above

16. If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$, then the relation between *u*, *v*, and *w* is :

(a) $v^2 = 2u + w$

(b) $v^2 = u + 2w$

(c) $u^2 = 2v + w$

(d) $u^2 = u + 2w$

17. Let ω be *k*-form of class C'' in some open set $E \subset R^n$, then :

(a) $d^2\omega = 0$

(b) $d^2\omega \neq 0$

(c) $d^2\omega = 1$

(d) none of the above

18. The positively oriented boundary of Q^2 is :

(a) $[e_1, e_2] + [0, e_2] + [0, e_1]$

(b) $[e_1, e_2] - [0, e_2] + [0, e_1]$

(c) $[e_1, e_2] + [0, e_2] - [0, e_1]$

(d) $[e_1, e_2] - [0, e_2] - [0, e_1]$

19. Let ω be a *k*-form in an open set $E \subset R^n$, then ω is said to be closed if :

(a) ω is of class C'

(b) ω is of class C' and $d\omega = 0$

(c) ω is of class C''

(d) ω is of class C'' and $d\omega = 0$

20. The unit square I^2 in R^2 is given by $I^2 = \sigma_1(Q^2) \cup \sigma_2(Q^2)$

where $\sigma_1(u) = u$ and $\sigma_2(u) = e_1 + e_2 - u$, then ∂I^2 is :

(a) $[0, e_1] + [e_1, e_1 + e_2] + [e_1 + e_2, e_2] + [e_2, 0]$

(b) $[0, e_1] + [e_1, e_1 + e_2] + [e_2, e_1, + e_2,] + [e_2, 0]$

(c) $[0, e_1] + [e_1, e_1 + e_2] + [e_2, e_1, + e_2,] + [0, e_2]$

(d) $[e_1, 0] + [e_1, e_1 + e_2] + [e_2, e_1 + e_2,] + [e_2, 0]$

Section—B $1\frac{1}{2}$ each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Write the statement of Weierstrass approximation theorem.
2. Define uniform convergence of a function.
3. Define power series.
4. Write the statement of Tauber's theorem.
5. Write the statement of inverse function theorem.
6. Write the statement of implicit function theorem.
7. Define extreme value of a function of several variables.

8. Define absolute maximum value for a real-valued function.
9. Define primitive mapping.
10. Define k-surface.

Section—C $2\frac{1}{2}$ each

(Short Answer Type Questions)

Note : Attempt all questions.

1. Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1 + nx^2}$ converges uniformly on $\mathbb{R}$.
2. Show that the series $\sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$ converges uniformly in $[1, \infty)$.
3. Prove that : $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots.,$ for $-1 \leq x \leq 1$.
4. Show that the series obtained by integrating a power series term by term has the same radius of convergence as the original series.
5. Let $A \in L(R^n, R^m)$, then prove that $\|A\| < \infty$ and A is a uniformly continuous mapping of R^n into R^m.
6. Let $A, B \in L(R^n, R^m)$, then prove that $\|A + B\| \leq \|A\| + \|B\|$.
7. Prove that $J . J' = 1$.

8. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.

9. Let E be an open set in R^n. T be a C'-mapping of E into an open set $V \subset R^m$. Let ω and λ be k-forms in V. Then prove that

$$(\omega + \lambda)_T = \omega_T + \lambda_T$$

10. Let E be an open set in R^n, T be a C'-mapping of E into an open set $V \subset R^m$. Let ω and λ be k-and m-forms in V. Then prove that :

$$(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$$

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. State and prove Cauchy's general principle of uniform convergence.

Or

Let $\{f_n\}$ be a sequence of functions, differentiable on [*a*, *b*] and such that $\{f_n(x_0)\}$ converges for some point x_0 on [*a*, *b*]. If $\{f_n'\}$ converges uniformly on [*a*, *b*], then show that $\{f_n\}$ converges uniformly on [*a*, *b*] to a function *f*, and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \, (a \le x \le b).$$

2. State and prove Abel's theorem for power series.

Or

If $\sum a_n$ is a series of complex number which converges absolutely then prove that every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

3. State and prove chain rule for a function of several variables.

Or

Prove that a linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

4. Find the shortest distance from the point $\left(\frac{3}{2}, 0\right)$ to the parabola $y^2 = 4x$.

Or

Suppose T is a C -mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, S is a C -mapping of V into an open set $W \subset R^p$ and ω is a *k*-form in W, so that ω_S is a k-form in V and both $(\omega_S)_T$ and ω_{ST} are *k*=forms in E, where ST is defined by (ST) (*x*) = S(T(*x*)). Then prove that

$$(\omega_S)_T = \omega_{ST}$$