Roll No. ....

X–293

M. A./M. Sc. (First Semester) EXAMINATION, Dec., 2013

MATHEMATICS

Paper Fourth

(Complex Analysis—I)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all questions. Attempt any *two* parts from each question/Unit. All questions carry equal marks.

Unit—I

1. (a) State and prove Cauchy's integral formula.

   (b) Let $f(z)$ be analytic in the region $|z| < \rho$ and let $z = re^{i\theta}$ be any point of this region. Then prove that :

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f\left(Re^{i\phi}\right)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi$$

   where R is any number such that $0 < R < \rho$.

   (c) Prove that the function $\sin\left[c\left(z + \frac{1}{z}\right)\right]$ can be expanded in a series of the type :

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

in which the coefficients of both z^n and z^{-n}, are :

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(2\,c\cos\theta)\cos n\,\theta\,d\,\theta$$

Unit—II

2. (a) Suppose the following :

   (i) $f(z)$ is analytic in a domain defined by $|z| < 1$,

   (ii) $|f(z)| < 1$

   (iii) $f(0) = 0$

   Then prove that :

$$|f(z)| \leq |z|$$

   and $|f'(z)| < 1$. Equality holds only if $f(z)$ is a linear transformation $w = ze^{i\alpha}$ where α is real constant.

   (b) Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C, and let $f(z) \neq 0$ on C. Then prove that :

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \mathrm{N} - \mathrm{P}$$

   where N is number of zeros and P is the number of poles inside C.

   (c) For the function $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ find the kind of singularity at $z = -2$.

Unit—III

3. (a) Define residue of a function $f(z)$ and discuss the residue of $f(z)$ at (i) simple pole at $z = a$ (ii) pole of order n at $z = a$ (iii) infinity.

(b) What do you mean by many valued function ? Why $\log z$ is a many valued function and prove that :

$$\int_0^\infty \frac{x^{1/6} \log x}{(1+x)^2}\, dx = 2\pi - \frac{\pi^2}{\sqrt{3}}$$

(c) Prove by contour integration :

$$\int_0^\infty \frac{\log(1+x^2)}{(1+x^2)}\, dx = \pi \log 2$$

Unit—IV

4. (a) Prove that the set of all bilinear transformations form a non-abelian group under the product of transformation.

(b) Find the fixed points and the normal form of the following transformation :

$$w = \frac{3iz+1}{z+i}$$

(c) Let $f(z)$ be an analytic function of z in a region D of the z-plane and $f'(z) \neq 0$ inside D. Then prove that the mapping $w = f(z)$ is conformal at the points of D.

Unit—V

5. (a) If $\{f_n\}$ is a sequence in H (G) and f belongs to C (G, C), such that $f_n \to f$ then prove that f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

(b) State and prove Montel's theorem.

(c) Let G be an open connected subset of C. Then prove that the following statements are equivalent :

(i) For any f in H (G) there is a sequence of polynomials that converge of f in H(G).

(ii) For any f in H (G) and any closed rectifiable curve γ and G,

$$\int_{\gamma} f = 0$$

(iii) Every function f in H (G) has a primitive.

(iv) For any f in H (G) such that $f(z) \neq 0$ for all z in G, there is a function g in H (G) such that $f(z) = \exp g(z)$.

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