Roll No. ....

B–310

M. A./M. Sc. (First Semester) EXAMINATION, Dec., 2017

MATHEMATICS

Paper First

(Advanced Abstract Algebra—I)

Time : Three Hours] [*Maximum Marks : 80*

[*Minimum Pass Marks : 16*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose one correct answer out of four alternative answers :

1. Let G be a group and $\{e\} = G_0 \subset G_1 \subset G_2 \subset ..... \subset G_r = G$ be a normal series in G. The series is said to be composition series, if :

   (a) Each G_i is maximal normal subgroup of G_{i+1}

   (b) $\frac{G_i}{G_{i-1}}$ is a simple group

   (c) Both (a) and (b) are true

   (d) None of the above

2. Let $f: 4Z \to Z_3$ defined by $f(4n) = [n]$ for all $4n \in 4Z$, then $\ker f$ is :
   (a) 8 Z
   (b) 12 Z
   (c) 3 Z
   (d) None of the above

3. Which of the following statement is false ?
   (a) Every finite group has a composition series
   (b) Composition series of a group is not necessarily unique
   (c) Both (a) and (b)
   (d) None of the above

4. A group G is said to be simple, if :
   (a) G has no proper normal subgroup
   (b) Only subgroups of G are $\{e\}$ and G
   (c) Both (a) and (b)
   (d) None of the above

5. Which of the following polynomial is reducible over Z ?
   (a) $2x^2 + 4$
   (b) $x^2 + 2x + 1$
   (c) Both (a) and (b)
   (d) None of the above

6. Let $f(x) \in Z[x]$ is said to be primitive, if :
   (a) it has prime coefficients
   (b) gcd of coefficients is 1
   (c) gcd of coefficient is > 1
   (d) None of the above

7. If $F \subset E \subset K$ be fields such that $[E:F]=m, [K:E]=n$, then :

(a) $[K:F]=mn$

(b) $[K:F]<\infty$

(c) Both (a) and (b)

(d) None of the above

8. $[Q(\sqrt{2},\sqrt{3}):Q(\sqrt{2}+\sqrt{3})]=$

(a) 0

(b) 1

(c) 2

(d) None of the above

9. Let $p(x) \in F[x]$ be minimal polynomial of $u \in E$, then which of the following statement is false ?

(a) $p(x)$ is monic

(b) $p(x)$ is irreducible

(c) $p(u)=0$

(d) None of the above

10. Let $u \in E$ is algebraic over F, where E is an extension of F, then which of the following statement is false ?

(a) $F(u)$ is finite extension of F

(b) $F(u)$ is algebraic extension of F

(c) Both (a) and (b)

(d) None of the above

11. Which of the following statement is false ?

(a) A finitely generated extension

(b) A finite extension need not be algebraic

(c) Both (a) and (b)

(d) None of the above

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12. Splitting field of $x^2 + 1 \in R[x]$ is :

    (a) Q

    (b) R

    (c) C

    (d) None of the above

13. Let E be a normal extension of F, then which of the following statement is true ?

    (a) E is splitting field of a family of polynomials in $F[x]$

    (b) Every irreducible polynomial in $F[x]$ that has a root in E has all its roots in E

    (c) Both (a) and (b)

    (d) All of the above

14. Let K be a splitting field of a polynomial $p(x) \in F[x]$ and α be a root and $p(x)$ such that $(x-\alpha)^5 \mid p(x)$ in $K[x]$, then the multiplicity of root α is :

    (a) ≥ 5

    (b) ≤ 5

    (c) $= 5$

    (d) None of the above

15. If $f(x)$ be an irreducible polynomial over a field of characteristic zero, then which of the following statement is false ?

    (a) $f(x)$ has no multiple roots

    (b) All roots of $f(x)$ has same multiplicity

    (c) Both (a) and (b)

    (d) None of the above

16. Which of the following statement is true ?
   (a) Prime field has no proper subfield
   (b) Every field contains a prime field
   (c) Both (a) and (b)
   (d) None of the above

17. Let H be a subgroup of G (E/F), then :
   (a) $H \subset G(E/E_H)$
   (b) $H = G(E/E_H)$
   (c) $H \supset G(E/E_H)$
   (d) None of the above

18. Which of the following statement is true ?
   (a) S_3 is nilpotent group
   (b) S_3 is solvable group
   (c) Both (a) and (b)
   (d) None of the above

19. Let $F \subset E \subset K$ be fields, then :
   (a) $G(E/F) \subset G(E/K)$
   (b) $G(E/K) \subset G(E/F)$
   (c) $G(K/F) \subset G(E/K)$
   (d) None of the above

20. Which of the following statement is false ?
   (a) Any algebraic extension of a field of char.0 is separable
   (b) Perfect field is separable
   (c) Algebraic extension of a finite field is separable
   (d) None of the above

Section—B $1\frac{1}{2}$ each

(Very Short Answer Type Questions)

Note : Answer in 2-3 sentences.

1. Define commutator subgroup.
2. Show that every abelian group is solvable.
3. Define center of a group.
4. Write *four* properties of $F[x]$.
5. Let $f(x) \in F\{x]$ be a polynomial of deg > 1. If $f(\alpha) = 0$ for some $\alpha \in F$, then show that $f(x)$ is reducible over F.
6. Define algebraic extension.
7. State Zorn's lemma.
8. What do you mean by characteristic of a field ?
9. Show that if E be a finite extension of a finite field F, then $E = F(\alpha)$ for some $\alpha \in E$.
10. Define transitive permutation group.

Section—C $2\frac{1}{2}$ each

(Short Answer Type Questions)

Note : Attempt all questions.

1. Show that every subgroup and every homomorphic image of a solvable group is solvable.
2. Find all composition series of Z/(30). Verify that they are equivalent.
3. Show that $f(x) = x^2 - x - 1 \in Z/(3)[x]$ is irreducible.
4. Let E be an extension field of F and let $u \in F$ be algebraic over F. Let $p(x) \in F[x]$ be a polynomial of the least degree such that $p(u) = 0$. Then show that :

   (a) $p(x)$ is irreducible over F.

   (b) If $g(x) \in F[x]$ such that $g(u) = 0$ then, $p(x) | g(x)$

5. Prove that if E is a finite extension of a field F, then E is an algebraic extension of F.
6. Find the degree of $Q(\sqrt{2}, \sqrt{3})$ over Q.
7. Let E be an algebraic extension of F and let $\sigma : E \to E$ be an embedding of E into itself over F. Then show that σ is onto and an automorphism of E.
8. Show that the multiplicative group of non-zero elements of a finite field is cyclic.
9. If K is the splitting field of $x^4 - x^2 + 1 \in Q[x]$, then find the Galois group G (K/Q).
10. Let E be splitting field of $x^n - a \in F[x]$. If F contains primitive *n*th root of unity, then show that G (E/F) is solvable.

Section—D

4 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. Prove that every subgroup and every homomorphic image of a nilpotent group is nilpotent.

Or

Prove that any *two* composition series of a finite group are equivalent.

2. Let E be an algebraic extension of a field F contained in some algebraic closure $\overline{F}$ or F. Then prove that the following conditions are equivalent :
   (a) Every irreducible polynomial in $F(x)$ that has a root in E splits into linear factors in E.
   (b) E is the splitting field of a family of polynomials in $F(x)$.
   (c) Every embedding σ of E into $\overline{F}$ that keeps each element of F fixed maps E onto E.

Or

Let $F \subset E \subset K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then show that :

(a) $[K:F] < \infty$

(b) $[K:F] = [K:E][E:F]$

3. Show that any prime field is either isomorphic to the field of rational numbers or to the field of integer modulo some prime.

Or

If $f(x) \in F[x]$ is irreducible over F, then show that all roots of $f(x)$ have the same multiplicity.

4. Let E and F be fields, show that the different embedding of F into E are linearly independent over F.

Or

Let E be a finite separable extension of field F. Prove that following are equivalent :

(a) E is normal extension of F

(b) F is fixed field of G (E/F)

5. Let $f(x)$ be polynomial over a field F with no multiple roots. Show that $f(x)$ is irreducible over F if and only if Galois group G of $f(x)$ is isomorphic to a transitive permutation group.

Or

Suppose that the field F has all nth root of unity and suppose that $a \neq 0$ is in F. Let $x^n - a \in F[x]$ and let K be its splitting field over F. Then show that :

(a) $K = F(u)$, where u is any root of $x^n - a$

(b) The Galois group of $x^n - a$ over F is abelian

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