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# A–309

## M. A./M. Sc. (First Semester) EXAMINATION, Dec., 2016

MATHEMATICS

Paper First

**(Advanced Abstract Algebra-I)**

*Time : Three Hours* ] [ *Maximum Marks : 80*

[*Minimum Pass Marks : 16*

**Note :** Attempt all Parts as directed.

**Part—A** 1 each

**(Objective Type Questions)**

**Note :** Choose the correct answer out of four alternative answers (a) through (d).

1. Let G be a group and $\{e\} = G_0 \subset G_1 \subset G_2 \subset \ldots\ldots \subset G_r = G$ be a chain of subgroups of G. The chain is said to subnormal series if, for all $i \in \{0, 1, \ldots\ldots r-1\}$ :
   - (a) Each $G_i$ is normal subgroup of $G_{i-1}$
   - (b) Each $G_i$ is normal subgroup of G
   - (c) Each $G_i$ is normal subgroup of $G_{i+1}$
   - (d) None of the above
2. Number of composition series in a simple group is :
   - (a) 0

(b) 1

(c) 2

(d) None of the above

3. Which of the following statement is not true ?

(a) Every abelian group is nilpotent

(b) Every nilpotent group is solvable

(c) Set of commutators of a group G itself form a semigroup of G.

(d) None of the above

4. $S_n$ is not nilpotent for :

(a) $n \geq 3$

(b) $n \geq 4$

(c) $n \geq 5$

(d) None of the above

5. Which of the following is not a property of F [$x$] ?

(a) F [$x$] is a UFD

(b) If $p(x)$ is irreducible in F [$x$], then $\frac{F[x]}{((p(x))}$ is a field

(c) F [$x$] is a PID

(d) None of the above

6. Let $f(x) \in Z[x]$ be a polynomial such that gcd of the coefficients of $f(x)$ is 1, then $f(x)$ is called :

(a) Irreducible

(b) Reducible

(c) Primitive

(d) None of the above

7. $\left[Q\left(\sqrt{3}, \sqrt{7}\right) : Q\left(\sqrt{3}\right)\right]$ is :

(a) 2

(b) 4

(c) 6

(d) None of the above

8. Let F be an extension field of K such that $[F : K] = 1$, then :

(a) $K \subseteq F$

(b) $K = F$

(c) $F \subsetneq K$

(d) None of the above

9. $Q\left(\sqrt{3}, -\sqrt{3}\right) =$

(a) $Q\left(\sqrt{3}\right)$

(b) $Q\left(\sqrt{0}\right)$

(c) $Q\left(2\sqrt{3}\right)$

(d) None of the above

10. Let $f(x) \in F[x]$ and K, L be two splitting fields of $f(x)$ over F, then :

(a) Either $K \subset L$ or $L \subset K$

(b) $K \approx L$

(c) $K = L$

(d) None of the above

11. The prime field of a field F is :
    - (a) Isomorphic to **Q**
    - (b) Isomorphic to **Z**/($p$), $p$ is prime
    - (c) Either (a) or (b)
    - (d) None of the above

12. Which of the following field is perfect ?
    - (a) Finite field
    - (b) Field of characteristics zero
    - (c) Both (a) or (b)
    - (d) None of the above

13. An extension E of a field F is said to be Galois extension of F, if :
    - (a) E is finite extension of F
    - (b) E is normal extension of F
    - (c) E is separable extension of F
    - (d) All of the above

14. If $p(x)$ be irreducible polynomial in F [$x$] and $\alpha, \beta$ are roots of $f(x)$ in some splitting field, then :
    - (a) $F(\alpha) \simeq \dfrac{F(x)}{(p(x))}$
    - (b) $F(\alpha) \simeq F(\beta)$
    - (c) Both (a) and (b) is true
    - (d) None of the above

15. If composition factors of a finite group are cyclic group of prime orders, then the group is :
    - (a) Solvable

(b) Nilpotent

(c) Both (a) and (b) is true

(d) None of the above

16. Let F contains primitive $n$th root of unity and $x^n - b \in F[x]$ be an irreducible polynomial, then the splitting field E of $f(x)$ is :

(a) Finite extension over F

(b) Cyclic extension over F

(c) Finite cyclic extension over F

(d) None of the above

17. Let E is finite extension of a field F, then :

(a) $|G(E/F)| = [E:F]$

(b) $|G(E/F)| \le [E:F]$

(c) $|G(E/F)| \ge [E:F]$

(d) None of the above

18. There is a system of $m$ homogeneous linear equations in $n$ unknowns, this system has a non-trivial solution, if :

(a) $m < n$

(b) $m = n$

(c) $m > n$

(d) None of the above

19. Set of all automorphism of a field under resultant composition form a :

(a) Field

(b) Group

(c) Ring

(d) None of the above

20. Let $f(x) \in F[x]$ and $E \subset K$ is splitting field of $f(x)$, such that $G(E/F) \simeq \dfrac{G(K/F)}{G(K/E)}$, then G (E/F) is solvable, if :

    (a) G (K/F) is solvable

    (b) G (K/E) is solvable

    (c) Both (a) and (b) is true

    (d) None of the above

## Part—B

### (Very Short Answer Type Questions)

**Note :** Answer in 2-3 sentences. $1\frac{1}{2}$ each

1. Write down condition for equivalence of two normal series $S = (G_0, G_1 \ldots\ldots\ldots, G_r)$ and $S' = (G'_0, G'_1, \ldots\ldots\ldots, G'_r)$.
2. What do you mean by class of nilpotency of a group $G'$ ?
3. Define irreducible polynomial.
4. Under what condition a finitely generated extension becomes a finite extension ?
5. Define algebraically closed field.
6. An irreducible polynomial over a field of characteristics 0 has how many multiple roots ?
7. Give *two* examples of a prime field.
8. When an irreducible polynomial $f(x) \in F[x]$ is called a separable polynomial ?
9. When we say two elements *a, b* of a field K are conjugate over a subfield F ?
10. Define solvability of a polynomial by radicals.

## Part—C

$2\frac{1}{2}$ each

### (Short Answer Type Questions)

**Note :** Attempt all questions.

1. Prove that every abelian group is solvable.
2. Show that :

$$\frac{4Z}{12Z} \simeq Z_3.$$

3. Let $f(x) \in F[x]$ be a polynomial of degree $> 1$. If $f(\alpha) = 0$, for some $\alpha \in F$, then show that $f(x)$ is reducible over F.
4. Prove that if E is a finite extension of a field F, then E is an algebraic extension of F.
5. Show that :

$$Q\left(\sqrt{2}, \sqrt{3}\right) = Q\left(\sqrt{2} + \sqrt{3}\right).$$

6. If $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, and $u = \cos\frac{2\pi}{n}$, then show that :

$$[Q(\omega) : Q(u)] = 2$$

7. Find the degree of the extension of the splitting field of $x^3 - 2 \in Q[x]$.
8. Let $f(x) \in F[x]$ be a polynomial of degree $\geq 1$ with $\alpha$ as a root. Show that $\alpha$ is a multiple root if and only if $f'(\alpha) = 0$.
9. If **K** is the splitting field of $x^4 - 3x^2 + 4 \in Q[x]$, then find Galois group G (**K/Q**).
10. Let E be a finite separable extension of a field F and let $H \subset G(E/F)$, then show that $G(E/E_H) = H$.

## Part—D 4 each

### (Long Answer Type Questions)

**Note :** Attempt all questions.

1. Prove that every subgroup and every homomorphic image of a nilpotent group is nilpotent.
2. Prove that for any field K the following are equivalent :
   (a) K is algebraically closed
   (b) Every irreducible polynomial in K $[x]$ is degree 1.
3. Let F be a field of characteristic 0. Prove that a finite separable extension E of F is a simple extension of F.
4. Let E be finite separable extension of a field F. Prove that the following statements are equivalent :
   (a) E is normal extension of F.
   (b) F is fixed field of G (E/F).
5. Let F be a field of characteristics 0 and E be splitting field $x^n - a \in F[x]$. Prove that G(E/F) is a solvable group.