Roll No. _

G-4013

M.Sc. FOURTH SEMESTER EXAMINATION 2014

MATHEMATICS

Paper – I

FUNCTIONAL ANALYSIS (II)

Time : 3 Hours

Max. Marks – 80

Note : Solve ***two*** parts from each question. Each question carries equal marks.

UNIT - I

1. a State and prove open mapping theorem.

   b. State and prove closed graph theorem.

   c. Let $\{T_n\}$ be a sequence of continuous linear operators of Banach Space X into Banach Space Y such that $\lim_{n\to\infty} T_n(x) = T(x)$ exists for every $x \in X$. Then prove that T is continuous linear operator and $\|T\| \leq \lim_{n\to\infty} \inf\|T_n\|$.

UNIT - II

2. a. Let X be a normed linear space over K and let S be a linear subspace of X. Suppose that $z \in X$ and $dist(z, X) = d > 0$. Then Prove that $\exists\ g \in X^*$, such that $g(s) = \{0\},\ g(z) = d\ and\ \|g\| = 1$.

   b. Prove that a Banach space is reflexive if and only if its dual space is reflexive.

   c. Let X and Y be normed spaces over the field K and $T : X \to Y$, a bounded linear operator, then

      (i) The adjoint $T^{\times}$ is bounded linear operator from Y^* to X^* i.e. $T^{\times} \in B(Y^*, X^*)$.

      (ii) $\|T^{\times}\| = \|T\|$

      (iii) The mapping $T \to T^*$ is an isometric isomorphism of $B(X,Y)$ into $B(Y^*, X^*)$.

UNIT - III

3. a. State and prove Cauchy-Schwartz inequality.
   b. In an Inner product Space, prove that x *and* y are orthogonal vectors if and only if

$$\|x + \alpha y\| = \|x - \alpha y\|, \forall \alpha \in K.$$

   c. Let $\{e_\alpha\}_{\alpha \epsilon \Lambda}$ be an orthonormal set in a Hilbert space H and $x \epsilon H$, then prove that , $x - \sum \langle x, e_\alpha \rangle e_\alpha \perp e_\beta$ for each $\beta \epsilon \Lambda$.

UNIT - IV

4. a. Let M be a closed subspace of a Hilberty Space H. Then prove that $H = M \oplus M^{\perp}$.
   b. Let y be a fixed vector in Hilbert Space H and let f_y be a scalar defined by $f_y(x) = \langle x, y \rangle \, \forall \, x \in H$. Show that $f_y \in B(H, K)$. Further show that, $\|y\| = \|fy\|$.
   c. Let H_1 *and* H_2 be Hilbert Spaces. $T, S \in B(H_1, H_2)$ and $\alpha \in K$, then prove that :
      i. $(T + S)^* = T^* + S^*$
      ii. $(\alpha T)^* = \bar{\alpha} T^*$
      iii. $(TS)^* = S^* T^*$
      iv. $\|T^* T\| = \|TT^*\|$

UNIT - V

5. a. Let H be a complex Hilbert Space. Then prove that, every $T \in B(H)$, can be expressed uniquely as $T = A + iB$, where $A, B \in S(H)$.
   b. Define normal operators. Let ℵ(H) be the set of all normal operators on H, then prove that:
      i. ℵ(H) is closed under scalar multiplication.
      ii. ℵ(H) is a closed subset of B(H).
   c. Define Unitary Operators. Let S *and* T be unitary operators on a Hilbert Space H, then prove that,
      i. T is isometric
      ii. T is Normal
      iii. ST is Unitary
      iv. T^{-1} is Unitary

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